Quiz Answer : Nicole's Wedge

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The trip home from her parent's house in Canton, Mississippi (32.61ºN 90.00ºW) would surely be exhausting. Nicole sat down for one last relaxing moment to recall how much she and her husband Andrew had enjoyed celebrating their anniversary while staying at her childhood home. As the long trip back to the real world faced them, Nicole's eyes caught sight of the sundial in the front yard, and she immediately decided to uproot it and bring it home with her. She had made the horizontal dial specifically for her latitude when she was still living in this old house; its location exactly on the 90^{th} meridian made it unnecessary to include any longitude correction. The gnomon, which rose from exactly the center point of the circular dial, was still erect and sturdy. She decided that the dial was not simply part of her past that she would visit occasionally. She wanted it now for her own front yard. It fit neatly into that one last space left in her luggage.

Now that Spring had clearly arrived, it was time to install the dial on her porch. Nicole waited for Andrew to return from the basement with the wooden wedge she had asked him to make to place under the dial in its new location. When Andrew arrived, the first thing she did was to test the angle of inclination of the wedge. As she expected, it came as close to matching her calculations as her instruments could detect.

The large top and bottom surfaces of the wedge were both square. The top square was just large enough for her to lay the circular plate of the dial on it, with four points of the dial plate just touching the four center points of the sides of the square.

She laid the dial on the top of the wedge so that it was neatly centered, just fitting exactly within the square designed to hold it. She turned the dial until the hour marking for slightly earlier than 8:25am was touching the top side of the wedge, and slightly later than 4:33pm was touching the right side. The calculations she had done in the plane on the way home had given her times of 8:24:40 and 16:33:20; so this was as close as she could come. She then secured the dial in its place so that it could not move on the wedge.

At last, she was ready to take the dial and wedge out onto the porch where it would be placed over the horizontal meridian line she had drawn several years ago as one of the first chores to be completed in a new house.

Nicole turned the wedge counterclockwise so that the side edge of its base made an angle with the meridian line exactly equal to the latitude of her house. Andrew was given the privilege of bolting it in place.

Now Nicole knew that the dial's gnomon was once again parallel to the celestial axis, and the time indicated on her dial would only need a correction for the equation of time for it to match the time on her watch throughout the next several months when she would be working in the yard. Her house now felt so much more like home.

Quiz:

Nicole and Andrew live on the outskirts of what U.S. state capital city?

What is the inclination of Nicole's wedge?

Nicole and Andrew live in Vermont, not far from the state capitol in Montpelier. The information given in the statement of the quiz, along with a quick referral to a U.S. map, is sufficient to solve for this portion of the answer. However, before discussing how to arrive at this result, let us first provide a general description of the wedge technique Nicole used to orient her sundial at its new location.

The Wedge Technique - Preliminaries

Suppose you have a horizontal sundial designed for latitude $L1$, with hour lines intended to indicate the local solar time at a site $M1$ degrees of longitude to the west or east of the dial's location. A traditional dial with no longitude correction will indicate local solar time at the dial's location; in this case, $M1 = 0°$.

If the hour lines on the dial are redrawn so that they include a longitude correction for Standard Time, then the lines indicate local solar time at the central meridian of the time zone. If that central meridian is, *e.g.*, 5° to the west of the dial's location, then $M1 = 5°$; if to the east, then $M1 = -5°$.

Finally, suppose that the dial with *M*1 = 5° has its hour lines relabeled to make a full hour's additional correction (*i.e.* 15°) for Daylight Saving Time. Then the dial will indicate local solar time for a location 10° to the east of the dial's actual longitude (since 5-15 = -10). With this relabelling, we would have *M*1 = -10°.

For the sake of a concrete example, suppose we have a horizontal dial at 30°N 85°W, with a longitude correction built into the layout of the hour lines to reflect the dial's being situated 5° degrees east of the central meridian (90°W) of its time zone. We thus have *L*1 = 30° and *M*1 = 5°. *M*1 is determined by the fact that the hour lines are laid out to indicate the local solar time at a location 5° of longitude to the west of the dial's actual site.

Now suppose we wish to move the dial to latitude 40°N 72°W and set it up so that it works properly, requiring only an Equation of Time correction to indicate Daylight Saving Time in its new location and time zone.

For the new location, we have *L*2 = 40° and *M*2 = -12°. The value of *M*2 comes from the fact that we wish to have the dial indicate the local solar time at a location 12° of longitude to the east of its new site (*i.e.* 60°W, which, for the sake of Daylight Saving Time, is 15° to the east of the new time zone's central meridian at 75°W).

Orienting The Dial Without A Wedge

If we use the information we now have, without introducing a wedge, we would orient the dial at the new location by the following method. Align the dial as though it were a normal horizontal sundial at the new location. Lift the front of the dial face up so that it inclines 10° (*i.e.* *L*2 - *L*1) above the North horizon. Now, holding the gnomon firmly, twist the entire dial around the shadow casting edge of the gnomon; do this so that the position in space of this shadow casting edge does not change, but the rest of the dial rotates around it. This rotation must be through an angle equal to –17° (i.e. *M*2 – *M*1). If we stand to the north of the dial so that the line from the center to the end of the gnomon points directly at us, then a negative value for this rotation corresponds to a clockwise turn – and a positive value to a counterclockwise turn.

This is not an easy process. It is difficult to hold everything in its proper position and to make the rotation while holding the gnomon at the proper angle. Nor is it easy to figure out how to anchor the dial once it has been manipulated as described.

To make the process easier physically (if not mathematically), we introduce a wedge.

The Wedge Technique – Design

The four values, *L*1, *M*1, *L*2 and *M*2 allow us to calculate 3 angles:

I = 17.0948° (inclination)
R = 49.6340° (rotation)
D = -59.4702° (declination)

The formulas for the calculations will be given below; for now, let us simply assume we have the needed values and proceed to the design of the wedge.

Inclination

To prepare the wedge, begin with a solid rectangular block of wood as shown in Fig 1. If the edges are true, it will rise vertically above a horizontal base. Cutting a solid block of wood may be difficult for some, but the concept can be applied to other approaches.

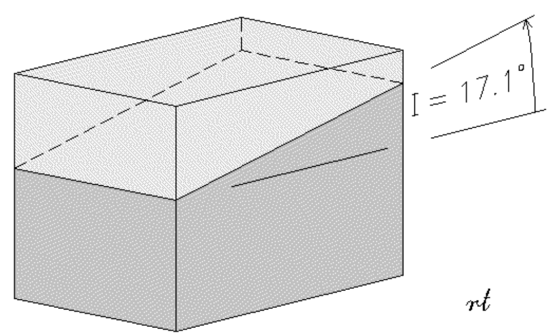

Figure 1.

Cut the top surface of the block so that it has a slope equal to the inclination angle I. This is the surface on which the dial will lie, so we in effect are changing the dial from a horizontal to one whose face is inclined above the horizontal plane by an angle I - in this example case, 17.0948°.

On the top surface, draw two lines parallel to the edges of the block and label them the X and Y axes as shown in Fig.2. If the block is on a level surface, you will note that a spirit level will only indicate level if held on, or parallel to, the X axis. The Y axis is the line sometimes referred to as the line of greatest slope.

Rotation

Now draw a line (the rotation line) through the intersection point of the coordinate axes, making an angle with the high end of the Y-axis equal to R, the rotation angle – in this example, 49.6340°. If R is positive, this angle should be measured clockwise from the axis; if negative, counterclockwise.

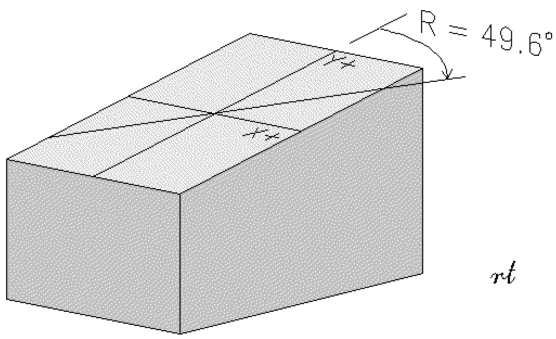

Figure 2.

Lay the dial on the top surface of the wedge such that its substilar line lies exactly on the Y-axis, with the north end of the substile at the high end of the axis. Now turn the dial (like rotating a knob) so that the north end of the stile rotates through an angle equal to the rotation angle and comes to rest exactly on top of the rotation line. Fix the dial in place on the wedge – it will not be moved anymore. Note however, that the wedge itself still needs to be movable so that it can be properly positioned.

Declination

There is one final maneuver needed to finish the project we have undertaken.

Draw a meridian (north-south) line on the horizontal surface on which you intend to rest the wedge. From the south end of this line, draw a second line at an angle equal to D (in this example, -59.4702°) from the meridian, with a positive angle corresponding to a clockwise measurement and a negative angle to a counterclockwise measurement.

Place the wedge so that the y-axis (or line of greatest slope) on its surface is directly over this new line. It may be easiest to accomplish this by extending the Y-axis line all the way to the edges on the top of the wedge and then dropping a plumb line to find the two points on the base of the wedge that should lie on the new line.

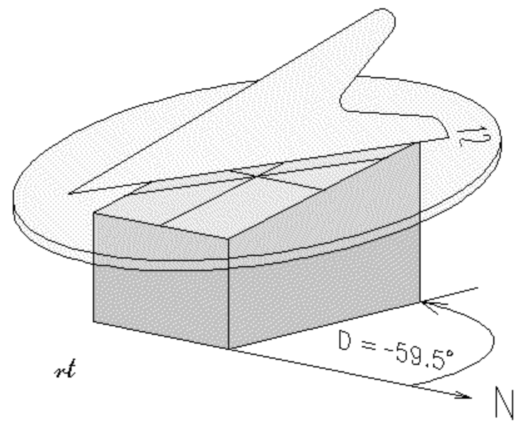

Figure 3.

Once this has been done, the wedge can be anchored in place. *The dial is now properly aligned*.

Calculation

To determine the required values for I, R and D, begin by calculating I from the following:

$$\cos I = \sin L1 \sin L2 + \cos L1 \cos L2 \cos(M2 - M1)$$
$$0^\circ \le I \le 90^\circ \qquad (1)$$

Now use this value of I to find R and D:

$$\cos R = \frac{\sin L2 - \cos I \sin L1}{\sin I \cos L1} \qquad (2)$$

$$\cos D = \frac{\cos I \sin L2 - \sin L1}{\sin I \cos L2} \qquad (3)$$

$$-180^\circ \le R \le 180^\circ \qquad -180^\circ \le D \le 180^\circ$$

Choose the signs of R and D so that the sign of R is the *opposite* of the sign of ($M2$- $M1$) and D has the *same* sign as ($M2$- $M1$).

If you fit $L1$, $L2$, $M1$ and $M2$ for the example dial into these equations, you will obtain the values for I, R and D as given above. You can also check for consistency – to be sure you have the correct values and signs – by fitting them into the following:

$$\frac{\sin I}{\sin(M2 - M1)} = \frac{\cos L2}{-\sin R} = \frac{\cos L1}{\sin D} \qquad (4)$$

<u>Derivation</u>

Rather than provide a great amount of detail on the derivation of these equations, I will point out that they are simple applications of the basic sine and cosine laws to the spherical triangle formed by drawing lines on the earth's surface, using the pole (P), the original site (Z1) and the new site (Z2) as vertices. In such a triangle, the sides and vertex angles are as given in the figure. When *L*1, *L*2, and *M*2-*M*1 are known, all the other elements of the triangle can be solved for – thus producing values for I, R and D . (For a more complete discussion of the required spherical trigonometry, see the author's articles "Solving the Spherical Triangle", *The Compendium*, Aug 1994, 1(3):8-10 and "Reducing A Plane To The Horizontal", *The Compendium*, Dec 1994, 1(4):19-23. See also the erratum on p.1 of the current issue.)

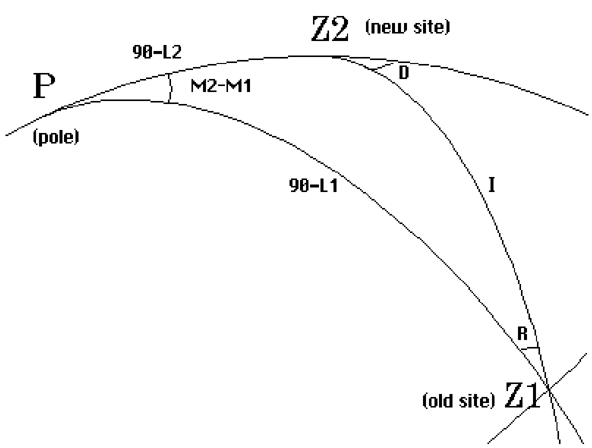

Figure 4.

This wedge technique is very general and can be used for the repositioning of most uprooted horizontal dials. Note that the dial does not need to be a classical dial with a gnomon parallel to the celestial axis. For example, an analemmatic dial can be moved from an original horizontal location to a new site using these same formulas. In this case, it is important to note that the gnomon must continue to be positioned perpendicular to the dial plate at the new site – so it will no longer be vertical – but the dial will work properly.

<u>Solving The Quiz</u>

What we have done so far is to provide a description of the method Nicole used to reposition her sundial at her new home. She was able to do this because she knew the locations of both sites and was familiar with the original design of her dial – she knew the values of *L*1, *M*1, *L*2, and *M*2. Unfortunately, as we try to solve the quiz, we find that we are not so lucky; that much information is not directly available to us.

We do know that $L1$ = 32.61º and that she lived on the time zone's central meridian. So, we also know that $M1$ = 0º, or, if her original design incorporated an adjustment for Daylight Saving Time, $M1$ = -15º; we will determine which of these values is the correct one in a moment.

Given her placement of the dial on the top of the wedge, we know she rotated it clockwise through a positive angle R, where:

$$\tan R = -\tan(t + M1)\sin 32.61^\circ \qquad (5)$$

and t=-53.8333º, corresponding to 8:24:40am.

Note however, that we should also check the value of the rotation with the given afternoon time. In this case we have,

$$\cot R = \tan(t + M1)\sin 32.61^\circ \qquad (6)$$

and t= 68.3333º, corresponding to 4:33:20pm.

We now have values for R as follows:

	If $M1$=0º	*If* $M1$=-15º
Eq. (5)	36.3990º	54.3032º
Eq. (6)	36.3966º	54.0995º
Average	36.3978º	54.2014º

The fact that the $M1$=-15º scenario produces relatively large differences in the values of R generated by the two equations should lead us to suspect that it is not the correct scenario; however, if we wait a moment, we will be able to provide positive proof for this suspicion.

If we now combine equation (4) with the given fact that L2 = - D and the trigonometric fact that for any angle $L2$, $\sin(2\cdot L2) = 2\cos L2 \sin L2$, we have:

$$\cos L1 \sin R = \cos L2 \sin L2 = 0.5\sin(2\cdot L2) \quad (7)$$

Now, since L1 and R are both positive, the maximum possible value of $\sin(2\cdot L2)$ is 1, so we can conclude that $\sin R \le 0.5/\cos L1 = 0.59357$ and therefore, $R \le 36.4108^\circ$. We therefore know that Nicole's original dial does not have a built-in adjustment for Daylight Saving Time, and $M1$=0º. Note, however, that a close reading of the quiz will show that she has decided to set the dial in its new location so that it will indicate Daylight Saving Time.

We now insert the value R = 36.3978º into equation (7) and we find that the latitude of Nicole's new home is $L2$ = 44.2874º.

At this point, we have values for two sides of the spherical triangle and two vertex angles – more than enough to solve the triangle completely. Readers may wish to finish the quiz by using the WEDGE computer program distributed with the digital edition of this issue of *The Compendium*.

Alternatively, keeping with the calculation method outlined in the articles noted above, we apply one of what are known as Napier's analogies to the triangle to obtain the formula:

$$\tan\frac{M2-M1}{2} = \tan\frac{D-R}{2}\sin\frac{L2-L1}{2}\Big/\cos\frac{L2+L1}{2}$$

We can now solve for $M2$ (knowing that $M1$ = 0º) to obtain $M2$ = -12.5913º (D and $M2$-$M1$ must have the same sign).

So, Nicole has set her sundial to indicate local solar time at a site 12.5913º of longitude to the east of her new home – and she has done so in order to account for Daylight Saving Time. Since setting the dial for DST means setting it for a location 15º to the east of her time zone's central meridian, we conclude the central meridian of Nicole's new time zone is 2.4087º (*i.e.* 15º - 12.5913º) of longitude west of her new home.

We now have enough information to track Nicole down. Of the various locations in the U.S. that fit with the data we have derived, only 44.2874ºN

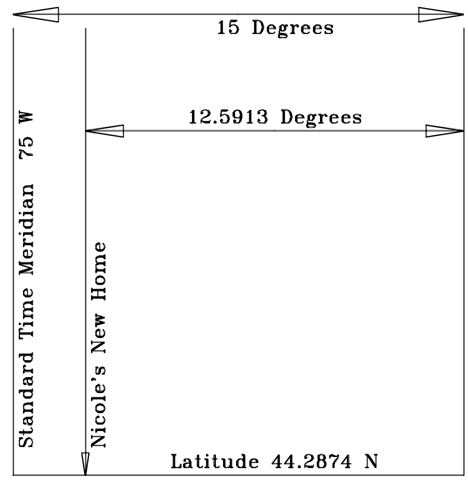

Figure 5.

72.5913ºW is anywhere near a state capital city; in fact, this location is within a few miles of the Vermont capitol building.

Finally, we have only to use the new found value for *M*2 - *M*1 along with values for either *L*1 and *D* or *L*2 and *R* in equation (4) to learn that the inclination *I* of Nicole's wedge is 15.2471º.

[*Bill Gottesman and Bill Buckler submitted answers to this quiz. Both found themselves within striking distance of Nicole's new location.*]

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One way to make the wedge.
An alternative to a block of wood - suggested by Bob Terwilliger, who also provided Figures 1-3.